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On the homology of the universal Steenrod algebra at odd primes

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Abstract We give an explicit description of the homology $H_*(\mathcal{Q})$ of the universal Steenrod algebra $\mathcal{Q}$ for any odd prime p, extending the work done for the $p = 2$ case. We also exhibit an isomorphism with a certain coalgebra of invariants Γ.

Keywords Universal Steenrod algebra · Invariant theory · Koszul algebras · Bar resolutions · Homology of algebras · Modular invariants

Mathematics Subject Classification 13A50 · 55S10

1 Introduction

Let p be any prime and $\mathbb{F}_p$ the field with p elements. In [14] May introduced the universal Steenrod algebra $\mathcal{Q}$ over the field $\mathbb{F}_p$ as the algebra of all cohomology operations in the category of H_∞-ring spectra. This algebra is also known as the *algebra of all generalized Steenrod operations* [13] or the *extended Steenrod algebra* [6]. It is closely related to other well known algebras, such as the opposite Λ^{opp} of the Λ algebra introduced in [1], the Steenrod algebra $\mathcal{A}$ in [15] and the Steenrod algebra for simplicial restricted Lie algebras A_L in [14]. The algebra $\mathcal{Q}$ has extensively been studied in [2–5,7,8] and [12]. In particular the papers [7] and [12] contain an invariant-theoretic description of $\mathcal{Q}$ and a computation of the diagonal cohomology $D^*(\mathcal{Q}) = \oplus Ext_{\mathcal{Q}}^{q,q}(\mathbb{F}_p, \mathbb{F}_p)$ for $p = 2$ and for any odd prime p, respectively. In [4] the authors prove that $\mathcal{Q}$, which is a non-locally finite homogeneous quadratic algebra, is an example of a *good* PBW-algebra, hence it is *koszul*, i.e. the cohomology $H^{*,*}(\mathcal{Q}) =$

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$\oplus Ext_{\mathcal{Q}}^{s,t}(\mathbb{F},\mathbb{F})$ is purely diagonal. The analogous for the homology $H_{*,*}(\mathcal{Q})$ holds: $Tor_{\mathcal{Q}}^{s,t}(\mathbb{F}_p,\mathbb{F}_p)=0$ if $s\neq t$. In [8] there is an explicit description of the non trivial part of the homology of $\mathcal{Q}$, $D_*(\mathcal{Q})=\oplus H_{q,q}(\mathcal{Q})$, for the $p=2$ case. The classical bar construction is used to get our main results: the mod p homology of $\mathcal{Q}$ and its description in terms of invariant theory. In Sect. 2 we recall the basic results of invariant theory. Section 3 is devoted to the computation of the diagonal homology $D_*(\mathcal{Q})$ of $\mathcal{Q}$. In Sect. 4 we prove that $D_*(\mathcal{Q})$ is isomorphic to the coalgebra Γ of certain invariants with respect to the action of the general linear group $GL(n,\mathbb{F}_p)$.

2 Preliminaries on invariant theory

Let $\Phi_n=(E(x_1,\ldots,x_n)\otimes\mathbb{F}_p[y_1,\ldots,y_n])[L_n^{-1}]$ be the localization out of the Euler class L_n of $H^*(B(\mathbb{Z}_p)^n)$. The general linear group $GL_n=GL(n,\mathbb{F}_p)$ acts on Φ_n. Let B_n be the Borel subgroup of GL_n of all upper triangular matrices and T_n the subgroup of B_n of all matrices with 1 on the main diagonal. We are interested on the invariant rings

$$\Delta_n=\Phi_n^{T_n},\quad \overline{\Delta}_n=\Phi_n^{B_n},\quad \Gamma_n=\Phi_n^{GL_n}.$$

We know from [9] (Corollary 1.3) that

$$\Delta_n=E(u_1,\ldots,u_n)\otimes\mathbb{F}_p[v_1^{\pm1},\ldots v_n^{\pm1}],$$

where u_i and v_i have degree $|u_i|=1$, $|v_i|=2$ for $1\le i\le n$.

Combining results from [11] and [7] (Proposition 1.2), we get

$$\overline{\Delta}_n=E(\overline{u}_1,\ldots,\overline{u}_n)\otimes\mathbb{F}_p[w_1^{\pm1},\ldots w_n^{\pm1}],$$

where

$$\begin{aligned}\overline{u}_i&=(-1)^iu_iv_i^{-1}, & w_i&=v_i^{p-1},\\ |\overline{u}_i|&=-1, & |w_i|&=2(p-1),\end{aligned}\tag{2.1}$$

for $1\le i\le n$.

In [9] (Corollary 1.2, item (ii)), we find

$$\Gamma_n=E(R_{n,0},\ldots,R_{n,n-1})\otimes\mathbb{F}_p[Q_{n,0}^{\pm1},Q_{n,1},\ldots,Q_{n,n-1}],$$

$|R_{n,i}|=2(p^n-p^i)-1$, $|Q_{n,i}|=2(p^n-p^i)$. Moreover, the following proposition gives recursive formulas for the generators of Γ_n.

Proposition 2.1 *(1)* $Q_{n,i}=Q_{n-1,i-1}^p+Q_{n-1,0}^{p-1}Q_{n-1,i}v_n^{p-1}=Q_{n-1,i-1}^p+Q_{n-1,0}^{p-1}Q_{n-1,i}w_n.$

(2) $R_{n,i} = Q_{n-1,0}^{p-1}(R_{n-1,i}v_n^{p-1} + Q_{n-1,i}u_n v_n^{p-2})$*; in terms of* B_n *invariants,* $R_{n,i} = Q_{n-1,0}^{p-1}(R_{n-1,i} + (-1)^n Q_{n-1,i}\overline{u}_n)w_n$.

Proof See Proposition 1.4 in [9]. □

By convention, $Q_{n,i} = 0$ for either $i < 0$ or $n < i$, $Q_{n,n} = 1$ and $R_{n,i} = 0$ for either $i < 0$ or $i \geq n$.

In particular we have

$$Q_{1,0} = v_1^{p-1} = w_1, \qquad R_{1,0} = u_1 v_1^{p-2} = -\overline{u}_1 w_1, \tag{2.2}$$

$$Q_{2,0} = Q_{1,0}^p v_2^{p-1} = w_1^p w_2,$$

$$Q_{2,1} = Q_{1,0}^p + Q_{1,0}^{p-1} v_2^{p-1} = w_1^{p-1}(w_1 + w_2),$$

$$R_{2,0} = Q_{1,0}^{p-1}(R_{1,0}v_2^{p-1} + Q_{1,0}u_2 v_2^{p-2}) = (\overline{u}_2 - \overline{u}_1)w_1^p w_2, \tag{2.3}$$

$$R_{2,1} = Q_{1,0}^{p-1} u_2 v_2^{p-2} = \overline{u}_2 w_1^{p-1} w_2.$$

Set

$$\Delta = \oplus_{n\geq 0}\Delta_n, \quad \overline{\Delta} = \oplus_{n\geq 0}\overline{\Delta}_n, \quad \Gamma = \oplus_{n\geq 0}\Gamma_n.$$

Here, $\Delta_0 = \overline{\Delta}_0 = \Gamma_0 = \mathbb{F}_p$.

For any non-negative integers n, q, t such that $q + t = n$, we define

$$\psi_{q,t} : \Delta_n \to \Delta_q \otimes \Delta_t$$

by setting

$$\psi_{q,t}(u_i) = \begin{cases} u_i \otimes 1, & 1 \leq i \leq q \\ 1 \otimes u_{i-q}, & q < i \leq n, \end{cases} \qquad \psi_{q,t}(v_i) = \begin{cases} v_i \otimes 1, & 1 \leq i \leq q \\ 1 \otimes v_{i-q}, & q < i \leq n. \end{cases} \tag{2.4}$$

The map $\psi_{q,t}$ turns out to be an isomorphism of algebras and Δ turns out to be a coalgebra with comultiplication $\psi : \Delta \to \Delta \otimes \Delta$ induced by the maps $\psi_{q,t}$ and defined by

$$\psi(\delta) = \sum_{q+t=n} \psi_{q,t}(\delta)$$

for any $\delta \in \Delta_n$.

Proposition 2.2 *For any* q, t, n *such that* $q + t = n$, $\psi_{q,t}(\overline{\Delta}_n) \subset \overline{\Delta}_q \otimes \overline{\Delta}_t$*, so* $\overline{\Delta}$ *is a subcoalgebra of* Δ.

Proof According to (2.1),

$$\psi_{q,t}(\overline{u}_i) = \begin{cases} \overline{u}_i \otimes 1, & 1 \leq i \leq q \\ 1 \otimes \overline{u}_{i-q}, & q < i \leq n, \end{cases} \qquad \psi_{q,t}(w_i) = \begin{cases} w_i \otimes 1, & 1 \leq i \leq q \\ 1 \otimes w_{i-q}, & q < i \leq n. \end{cases} \tag{2.5}$$

□

Proposition 2.3 *For any q, t, n such that $q + t = n$, the following relations hold*

(1) $\psi_{q,t}(Q_{n,i}) = \sum_{j\geq 0} Q_{q,0}^{p^t-p^j} Q_{q,i-j}^{p^j} \otimes Q_{t,j}$;
(2) $\psi_{q,t}(R_{n,i}) = Q_{q,0}^{p^t-1} R_{q,i} \otimes Q_{t,0} + \sum_{j\geq 0} Q_{q,0}^{p^t-p^j} Q_{q,i-j}^{p^j} \otimes R_{t,j}$.

Proof See Proposition 3.3 in [9]. □

Corollary 2.4 $\psi(\Gamma) \subset \Gamma \otimes \Gamma$*, so* Γ *is a subcoalgebra of* $\overline{\Delta}$ *and of* Δ.

Proof For any $f \in \Gamma_n$, $\psi(f) = \sum_{q+t=n} \psi_{q,t}(f)$ belongs to $\Gamma_q \otimes \Gamma_t$, so the restriction to Γ of the comultiplication $\psi : \Delta \to \Delta \otimes \Delta$ defines a comultiplication on Γ. □

3 The homology of $\mathcal{Q}$ over $\mathbb{F}_p$

The universal Steenrod algebra $\mathcal{Q}$ at odd primes is generated as an $\mathbb{F}_p$-algebra by

$$\mathcal{F} = \{ z_{\varepsilon,i} \mid \varepsilon \in \{0, 1\},\ i \in \mathbb{Z} \} \cup \{1\} \quad \text{with} \quad \deg z_{\varepsilon,i} = 2i(p-1) + \varepsilon,$$

subject to the following generalized Adem relations:

$$z_{\varepsilon,pk-1-n} z_{0,k} = \sum_j A_{(n,j)}\, z_{\varepsilon,pk-1-j} z_{0,k-n+j}, \tag{3.1}$$

$$z_{1-\varepsilon,pk-n} z_{1,k} = \sum_j A_{(n,j)}\, z_{1-\varepsilon,pk-j} z_{1,k-n+j} + \varepsilon \sum_j B_{(n,j)}\, z_{1,pk-j} z_{0,k-n+j}, \tag{3.2}$$

for each $(k, n) \in \mathbb{Z} \times \mathbb{N}_0$, where $A_{(n,j)}$ and $B_{(n,j)}$ are respectively equal to

$$(-1)^{j+1}\binom{(p-1)(n-j)-1}{j} \text{ and } (-1)^j\binom{(p-1)(n-j)}{j}.$$

Such presentation already appeared in [7] where the authors also proved that

$$\mathcal{B} = \{z_{\varepsilon_1,i_1} \ldots z_{\varepsilon_h,i_h} \mid i_j \geq p i_{j+1} + \varepsilon_{j+1} \text{ for each } \ j = 1, \ldots, h-1\} \cup \{1\}$$

is a basis of $\mathcal{Q}$, called the basis of *admissible* monomials.

Let T denote the associative algebra freely generated by the $\mathbb{F}_p$-module with basis $\mathcal{F}$. We consider the map $d : T \to T$ given by $d(z_{\varepsilon,i}) = z_{\varepsilon,i-1}$, such that $d(\tau_1\tau_2) = d(\tau_1)\tau_2 + \tau_1 d(\tau_2)$ for any $\tau_1, \tau_2 \in T$. Then d is a derivation in T. We write d^s for the s-iterated of d. Let L be the two-sided ideal generated by the set

$$\{d^s(z_{0,ph-1}z_{0,h}),\ d^s(z_{1,ph-1}z_{0,h}),\ d^s(z_{0,ph}z_{1,h} - z_{1,ph}z_{0,h}),\ d^s(z_{1,ph}z_{1,h})\}$$

for all $s \in \mathbb{N}_0$, $h \in \mathbb{Z}$.

Proposition 3.1 *The algebras $\mathcal{Q}$ and T/L are isomorphic.*

Proof $\mathcal{Q}$ and T/L are both isomorphic to $\overline{\Delta}/(\Gamma_2)$; the isomorphisms are established by Proposition 2.5 and Proposition 2.3 of [7], respectively. □

If $z_{\varepsilon_1,i_1}\dots z_{\varepsilon_h,i_h}\in\mathcal{B}$, the string $I=((\varepsilon_1,i_1),(\varepsilon_2,i_2),\dots,(\varepsilon_h,i_h))\in(\{0,1\}\times\mathbb{Z})^h$ will be called the *label* of $z_{\varepsilon_1,i_1}\dots z_{\varepsilon_h,i_h}$ and we write z_I instead of $z_{\varepsilon_1,i_1}\dots z_{\varepsilon_h,i_h}$. We say that z_I has *length h* and *total degree* $\varepsilon_1+\dots+\varepsilon_h+2(p-1)(i_1+\dots+i_h)$; hence $\mathcal{Q}$ is a bigraded algebra. It is also an augmented algebra through the map $\varepsilon:\mathcal{Q}\to\mathbb{F}_p$ which vanishes on the monomials of positive length and is the identity over $\mathbb{F}_p\subset\mathcal{Q}$. Let us denote by J the augmentation ideal $J=\ker(\varepsilon)$.

Let $\overline{B}(\mathcal{Q})=T(J)=\oplus_{s\in\mathbb{N}_0}J\otimes\dots\otimes J$. Thus $\overline{B}(\mathcal{Q})$ is generated by elements of the form $z_{I_1}\otimes\dots\otimes z_{I_s}$ where $z_{I_j}\in J$. Such elements are written simply as

$$[z_{I_1}|\cdots|z_{I_s}]=[z_{\varepsilon_1,i_1}\dots z_{\varepsilon_{t_1},i_{t_1}}|z_{\varepsilon_{t_1+1},i_{t_1+1}}\dots z_{\varepsilon_{t_2},i_{t_2}}|\cdots|z_{\varepsilon_{t_{s-1}+1},i_{t_{s-1}+1}}\dots z_{\varepsilon_{t_s},i_{t_s}}]$$

and are trigraded: s is the *homological degree*, $t=t_s$ is the *length* and $d=2(p-1)\sum_{k=1}^{t_s}i_k+\sum_{k=1}^{t_s}\varepsilon_k$ is the *total degree*, which we usually disregard in notations. Let $\overline{B}_s(\mathcal{Q})_t$ be the submodule generated by elements of bidegree (s,t). Given a generator $z=[z_{I_1}|\cdots|z_{I_s}]$ of $\overline{B}_s(\mathcal{Q})$, for any $j=1,\dots,s-1$, let

$$\partial_{s,j}:\overline{B}_s(\mathcal{Q})\to\overline{B}_{s-1}(\mathcal{Q})$$

be the map defined by

$$\partial_{s,j}(z)=[z_{I_1}|\cdots|z_{I_j}z_{I_{j+1}}|\cdots|z_{I_s}].$$

Then we consider the following differential ∂ for $\overline{B}(\mathcal{Q})$:

$$\partial_s:\overline{B}_s(\mathcal{Q})\to\overline{B}_{s-1}(\mathcal{Q})$$

defined by

$$\partial_s(z)=\sum_{j=1}^{s-1}(-1)^{e_{I_j}}\partial_{s,j}(z),$$

where $e_{I_j}=j+\sum_{k=1}^{j}|z_{I_k}|$, being $|z_{I_k}|$ the total degree of z_{I_k}. The chain complex $(\overline{B}(\mathcal{Q}),\partial)$, known as the *reduced bar construction*, computes the homology of $\mathcal{Q}$, $H_{s,t}(\mathcal{Q})=\mathrm{Tor}_{s,t}^{\mathcal{Q}}(\mathbb{F}_p,\mathbb{F}_p)$. We know by [4] that $\mathrm{Tor}_{s,t}^{\mathcal{Q}}(\mathbb{F}_p,\mathbb{F}_p)=0$ when $s\neq t$, so we are only interested on the diagonal part of the homology:

$$D_*(\mathcal{Q})=\oplus_{k\geq 0}D_k(\mathcal{Q})=\oplus_{k\geq 0}H_{k,k}(\mathcal{Q}).$$

The group $D_k(\mathcal{Q})$ turns out simply to be $\ker(\partial_k):\overline{B}_k(\mathcal{Q})_k\to\overline{B}_{k-1}(\mathcal{Q})_k$, since there exist no non-zero $(k+1)$-chains of length k. The following Theorem helps to identify the elements

$$z = \sum_I f_I [z_{\epsilon_1, i_1} | \cdots | z_{\epsilon_k, i_k}] \tag{3.3}$$

in $\overline{B}_k(Q)_k$ belonging to $D_k(Q) = \ker(\partial_k)$. Note that only a finite number of $\mathbb{F}_p$-coefficients f_I in (3.3) is non-zero.

Theorem 3.2 *The element $z = \sum_I f_I [z_{\epsilon_1, i_1} | \cdots | z_{\epsilon_k, i_k}] \in \overline{B}_k(Q)_k$ is a cycle if and only if for each $j (1 \leq j \leq k-1)$ and each $((\epsilon_1, s_1), \ldots, (\epsilon_{j-1}, s_{j-1})) \in (\{0, 1\} \times \mathbb{Z})^{j-1}$, $((\epsilon_{j+2}, s_{j+2}), \ldots, (\epsilon_k, s_k)) \in (\{0, 1\} \times \mathbb{Z})^{k-j-1}$, the following condition holds:*

$$\sum_I f_I z_{e_j, i_j} z_{e_{j+1}, i_{j+1}} = 0,$$

where the summation runs over all I's such that

$$((e_1, i_1), \ldots (e_{j-1}, i_{j-1})) = ((\epsilon_1, s_1), \ldots, (\epsilon_{j-1}, s_{j-1}))$$

and

$$((e_{j+2}, i_{j+2}), \ldots (e_k, i_k)) = ((\epsilon_{j+2}, s_{j+2}), \ldots, (\epsilon_k, s_k)).$$

Proof One can follow the same argument used to prove Theorem 1 in [8]. □

As a consequence of this result we have the following Corollary. It can be proved by an argument similar to that for $p = 2$ in Corollary 2 of [8].

Corollary 3.3 *Suppose that $z = \sum_I f_I [z_{\epsilon_1, i_1} | \cdots | z_{\epsilon_k, i_k}] \in \overline{B}_k(Q)_k$ is a cycle. For each $S = ((e_1, s_1), \ldots, (e_q, s_q)) \in (\{0, 1\} \times \mathbb{Z})^q$ and $S' = ((e_{q+1}, s_{q+1}), \ldots, (e_k, s_k)) \in (\{0, 1\} \times \mathbb{Z})^{k-q}$, let z_S equal to*

$$\sum_I f_I [z_{\epsilon_{q+1}, i_{q+1}} | \cdots | z_{\epsilon_k, i_k}],$$

where the summation runs over the labels I such that

$$(\epsilon_1, i_1) = (e_1, s_1), \ldots, (\epsilon_q, i_q) = (e_q, s_q),$$

and $z_{S'}$ equal to

$$\sum_I f_I [z_{\epsilon_1, i_1} | \cdots | z_{\epsilon_q, i_q}],$$

where the summation runs over the labels I such that

$$(\epsilon_{q+1}, i_{q+1}) = (e_{q+1}, s_{q+1}), \ldots, (\epsilon_k, i_k) = (e_k, s_k).$$

Then z_S is a cycle of $\overline{B}_{k-q}(Q)_{k-q}$ and $z_{S'}$ is a cycle of $\overline{B}_q(Q)_q$.

Proposition 3.4 *Set*

$$R(k,n,\varepsilon)=[z_{\varepsilon,pk-1-n}|z_{0,k}]-\sum_j A_{(n,j)}\,[z_{\varepsilon,pk-1-j}|z_{0,k-n+j}]$$

and

$$S(k,n,\varepsilon)=[z_{1-\varepsilon,pk-n}|z_{1,k}]-\sum_j A_{(n,j)}\,[z_{1-\varepsilon,pk-j}|z_{1,k-n+j}]$$
$$+\varepsilon\sum_j B_{(n,j)}\,[z_{1,pk-j}|z_{0,k-n+j}].$$

Then $D_2(\mathcal{Q})$ has $\{R(k,n,\varepsilon),\ S(k,n,\varepsilon)\}_{k\in\mathbb{Z},n\in\mathbb{N}_0,\varepsilon\in\{0,1\}}$ as a linear $\mathbb{F}_p$-basis.

Proof To see this, observe that

$$\partial_2([z_{\varepsilon_1,i_1}|z_{\varepsilon_2,i_2}])=(-1)^{1+\varepsilon_1+2i_1(p-1)}[z_{\varepsilon_1,i_1}z_{\varepsilon_2,i_2}]=(-1)^{1+\varepsilon_1}[z_{\varepsilon_1,i_1}z_{\varepsilon_2,i_2}].$$

Then $\partial_2(R(k,n,\epsilon))$ and $\partial_2(S(k,n,\epsilon))$ vanish since they correspond to the generating relations (3.1) and (3.2) of $\mathcal{Q}$. □

Now we give some examples of cycles constructed by iterating the following generalized Adem relations:

$$z_{0,pk-1}z_{0,k}=0,\quad z_{1,pk}z_{1,k}=0,\quad z_{1,pk-1}z_{0,k}=0.$$

They are

$$z=[z_{0,p^{m-1}k-\frac{p^{m-1}-1}{p-1}}|z_{0,p^{m-2}k-\frac{p^{m-2}-1}{p-1}}|\cdots|z_{0,pk-1}|z_{0,k}]$$

and

$$z'=[z_{1,p^{m-1}k}|[z_{1,p^{m-2}k}|\cdots|z_{1,pk}|z_{1,k}],$$

both elements of $D_m(\mathcal{Q})$.

Further, for any $1\le j<m$, we get another element of $D_m(\mathcal{Q})$ given by the following chain:

$$z''_j=[z_{1,\alpha_{m-1}}|\cdots|z_{1,\alpha_{j+1}}|z_{1,\alpha_j}|z_{0,\alpha_{j-1}}|\cdots|z_{0,\alpha_1}|z_{0,\alpha_0}],$$

where

$$\alpha_t=\begin{cases}p^tk-\frac{p^t-1}{p-1} & \text{if}\quad 0\le t\le j-1\\ p^tk-p^{t-j+1}\frac{p^{j-1}-1}{p-1} & \text{if}\quad j\le t\le m-1\end{cases}\tag{3.4}$$

Then, using the Adem relation $z_{0,pk}z_{1,k}=z_{1,pk}z_{0,k}$ in addition to the others above, we get the following cycle of $D_3(\mathcal{Q})$:

$$z_3 = [z_{0,p^2k}|z_{1,pk}|z_{1,k}] + [z_{1,p^2k}|z_{0,pk}|z_{1,k}] + [z_{1,p^2k}|z_{1,pk}|z_{0,k}]$$

and two examples of cycles in $D_4(\mathcal{Q})$:

$$\begin{aligned} z_4 &= [z_{\epsilon,p^3k-1}|z_{0,p^2k}|z_{1,pk}|z_{1,k}] + [z_{1,p^3k}|z_{1,p^2k}|z_{0,pk}|z_{1,k}] \\ &\quad + [z_{1,p^3k}|z_{1,p^2k}|z_{1,pk}|z_{0,k}], \\ z_4' &= [z_{1,p^3k}|z_{0,p^2k}|z_{1,pk}|z_{1,k}] + [z_{1,p^3k}|z_{1,p^2k}|z_{0,pk}|z_{1,k}] \\ &\quad + [z_{1,p^3k}|z_{1,p^2k}|z_{1,pk}|z_{0,k}] + [z_{0,p^3k}|z_{1,p^2k}|z_{1,pk}|z_{1,k}]. \end{aligned}$$

Theorem 3.5 *The diagonal homology $D_*(\mathcal{Q})$ has a coalgebra structure given by*

$$\begin{aligned} &\psi_n : D_n(\mathcal{Q}) \to \oplus_{q+t=n}(D_q(\mathcal{Q}) \otimes D_t(\mathcal{Q})), \\ &z = \sum_I f_I[z_{\varepsilon_1,i_1}|z_{\varepsilon_2,i_2}|\cdots|z_{\varepsilon_n,i_n}] \mapsto z \otimes 1 + 1 \otimes z + \sum z' \otimes z'', \end{aligned}$$

where the cycles z' and z'' are obtained by splitting all the summands of z and suitably grouping the common terms.

Proof According to Corollary 3.3, the elements z' and z'', coming from the procedure described in the statement above, are cycles. □

4 The isomorphism between $D_*(Q)$ and Γ

We want to show that the diagonal homology of $\mathcal{Q}$ is isomorphic to Γ. To this purpose, let us consider the $\mathbb{F}_p$-linear maps $\pi_{n,q} : \overline{\Delta}_n \to \overline{B}_{n-1}(\mathcal{Q})_n$, for $n \geq 2$ and $q = 1, \ldots, n-1$, defined as follows: given $\overline{u}^{\mathcal{E}} w^I = \overline{u}_1^{\varepsilon_1} \ldots \overline{u}_n^{\varepsilon_n} w_1^{i_1} \ldots w_n^{i_n} \in \overline{\Delta}_n$,

$$\pi_{n,q}(\overline{u}^{\mathcal{E}} w^I) = [z_{1-\varepsilon_1,i_1}|\cdots|z_{1-\varepsilon_{q-1},i_{q-1}}|z_{1-\varepsilon_q,i_q} z_{1-\varepsilon_{q+1},i_{q+1}}|\cdots|z_{1-\varepsilon_n,i_n}].$$

We begin by looking at the map $\pi_{2,1}$.

Proposition 4.1 *ker $\pi_{2,1} = \Gamma_2$.*

Proof The map $\pi_{2,1} : \overline{\Delta}_2 \to \overline{B}_1(\mathcal{Q})_2$ acts as follows:

$$\pi_{2,1}(\overline{u}_1^{\epsilon_1} \overline{u}_2^{\epsilon_2} w_1^{i_1} w_2^{i_2}) = [z_{1-\varepsilon_1,i_1} z_{1-\varepsilon_2,i_2}].$$

Using relations (2.3) and the linearity of $\pi_{2,1}$, we get

$$\begin{aligned} \pi_{2,1}(Q_{2,0}^k Q_{2,1}^s) &= \pi_{2,1}\left(w_1^{pk+(p-1)s} w_2^k (w_1 + w_2)^s\right) \\ &= \pi_{2,1}\left(\sum_{j=0}^{s} \binom{s}{j} w_1^{p(k+s)-s+j} w_2^{k+s-j}\right) \end{aligned}$$

$$= \sum_{j=0}^{s} \binom{s}{j} \pi_{2,1}\left(w_1^{p(k+s)-s+j} w_2^{k+s-j}\right)$$
$$= \sum_{j=0}^{s} \binom{s}{j} [z_{1,p(k+s)-(s-j)} z_{1,k+s-j}]$$
$$= [d^s(z_{1,p(k+s)} z_{1,k+s})],$$

and $d^s(z_{1,p(k+s)} z_{1,k+s}) = 0$ according to Proposition 3.1. In a similar way, one can also prove that

$$\pi_{2,1}(R_{2,0}R_{2,1}Q_{2,0}^k Q_{2,1}^s) = -[d^s(z_{0,p(k+s+2)-1} z_{0,k+s+2})] = 0,$$
$$\pi_{2,1}(R_{2,1}Q_{2,0}^k Q_{2,1}^s) = [d^s(z_{1,p(k+s+1)-1} z_{0,k+s+1})] = 0,$$
$$\pi_{2,1}(R_{2,0}Q_{2,0}^k Q_{2,1}^s) = [d^s(z_{1,p(k+s+1)} z_{0,k+s+1} - z_{0,p(k+s+1)} z_{1,k+s+1}] = 0,$$

that is the elements of $\Gamma_2 \subset \overline{\Delta}_2$ correspond to the defining relations of $\mathcal{Q}$ in terms of d. Hence Γ_2 is the kernel of $\pi_{2,1}$. □

Lemma 4.2 *For any $n \geq 2$ and $q = 1, \ldots, n-1$:*

$$ker\, \pi_{n,q} = \overline{\Delta}_{q-1} \otimes \Gamma_2 \otimes \overline{\Delta}_{n-q-1}.$$

Proof For any $t \in \mathbb{N}$, we define $\pi_t : \overline{\Delta}_t \to \overline{B}_t(\mathcal{Q})_t$ as

$$\pi_t(\overline{u}_1^{\varepsilon_1} \ldots \overline{u}_t^{\varepsilon_t} w_1^{i_1} \ldots w_t^{i_t}) = [z_{1-\varepsilon_1,i_1} | \cdots | z_{1-\varepsilon_t,i_t}].$$

We write f for the composition $(\psi_{q-1,2} \otimes 1) \circ \psi_{q+1,n-q-1}$:

$$f : \overline{\Delta}_n \to \overline{\Delta}_{q-1} \otimes \overline{\Delta}_2 \otimes \overline{\Delta}_{n-q-1}.$$

Then

$$\pi_{n,q} = (\pi_{q-1} \otimes \pi_{2,1} \otimes \pi_{n-q-1}) \circ f.$$

Our result follows from Proposition 4.1 and the fact that f, π_{q-1} and π_{n-q-1} are $\mathbb{F}_p$-linear isomorphisms. □

Lemma 4.3 *The general linear group $GL_n(\mathbb{F}_p)$ is generated by all matrices of the form*

$$M = \begin{pmatrix} I_{q-1} & O & O \\ O & A & O \\ O & O & I_{n-q-1} \end{pmatrix},$$

where $A \in GL_2(\mathbb{F}_p)$.

Proof This result follows from the fact that every invertible matrix admits an *elementary bidiagonal factorization* (see [10]). □

According to the previous Lemma,

$$\Gamma_n = \cap_{q=1}^{n-1} \overline{\Delta}_{q-1} \otimes \overline{\Gamma}_2 \otimes \overline{\Delta}_{n-q-1}.$$

Combining with Lemma 4.2, we arrive at

$$\Gamma_n = \cap_{q=1}^{n-1} \ker \pi_{n,q}. \tag{4.1}$$

We write h_n for the $\mathbb{F}_p$-linear isomorphism inverse to π_n,

$$h_n = \pi_n^{-1} : \overline{B}_n(\mathcal{Q})_n \to \overline{\Delta}_n, \quad h_n([z_{\varepsilon_1,i_1}|\cdots|z_{\varepsilon_n,i_n}]) = \overline{u}_1^{1-\varepsilon_1} \ldots \overline{u}_n^{1-\varepsilon_n} w_1^{i_1} \ldots w_n^{i_n}.$$

We observe that, for any $q = 1, \ldots, n-1$, $\partial_{n,q} : \overline{B}_n(\mathcal{Q})_n \to \overline{B}_{n-1}(\mathcal{Q})_n$ is the result of the composition $\pi_{n,q} \circ h_n$. We are going to use this fact in the proof of our main result.

Theorem 4.4 Γ *and* $D_*(\mathcal{Q})$ *are isomorphic as coalgebras.*

Proof The maps $\{ h_n \}_{n\in\mathbb{N}}$ establish a map of coalgebras

$$h : \oplus_{n\in\mathbb{N}} \overline{B}_n(\mathcal{Q})_n \to \overline{\Delta}.$$

A chain $z \in \overline{B}_n(\mathcal{Q})_n$ represents a cycle if and only if $\partial_{n,q}(z) = (\pi_{n,q} \circ h_n)(z) = 0$ for any $q = 1, \ldots, n-1$. This holds if and only if $h_n(z) \in \cap_{q=1}^{n-1} \ker \pi_{n,q}$, that is $h_n(z) \in \Gamma_n$ according to (4.1). Then h_n restricts to an isomorphism of coalgebras

$$\overline{h}_n : D_n(\mathcal{Q}) \to \Gamma_n.$$

□

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